Exam
Random Geometry and Topology B

3 November 2025

- You have from 11.45 until 13.45.
- It is not allowed to use phones, computers, books, notes or any other aids.
- **Write each exercise on a separate sheet of paper**, with your name and student number clearly legible on each sheet.

Consider the **contact-voter model** $(\xi_t)_{t\geq 0}$ with rate $\lambda > 0$ on the integer line $\mathbb{Z}$. This process is a mixture of a rate λ contact process and a voter model. That is, we start the process from an initial configuration of healthy (state 0) and infected (state 1) sites at $t = 0$. If a site is infected, then it becomes healthy after an exponential time with parameter 1 independently of the state of the other sites. If a site is healthy, it gets infected from each of its infected neighbours after an exponential time with parameter λ (that is, with mean $1/\lambda$). Finally, each site (regardless of their state) chooses a neighbour uniformly at random after an exponential time with parameter 1 and adopts its state.

We will abuse notation and identify a configuration $\eta \in S = \{0, 1\}^{\mathbb{Z}}$ with the set of vertices in state 1 in η.

Exercise 1 (20 pts)
Write down the generator of $(\xi_t)_{t\geq 0}$. Is the process monotone? Why/why not?

Exercise 2 (20 pts)
Describe a graphical representation of the process and define ξ_t, the set of vertices in state 1 at some time $t > 0$, in terms of this representation.

Exercise 3 (20 pts)
Denote by ξ_t^0 the state of the process at time t starting from the initial configuration $\xi_0^0 = \{0\}$ and define the critical parameter

$$\lambda_c := \inf\{\lambda : \mathbb{P}(\xi_t^0 \neq \varnothing \ \forall t \geq 0) > 0\}.$$

Prove that $\lambda_c < \infty$.

(In the solution bounds for the critical parameters of other processes proven in the lectures or tutorials may be used without proof.)

Exercise 4 (20 pts)
Find a dual process for $(\xi_t)_{t\geq 0}$ with respect to the duality function $H : S \times \tilde{S} \to \{0, 1\}$

$$H(\xi, \tilde{\xi}) = \mathbb{1}\{\xi \cap \tilde{\xi} = \varnothing\},$$

where $\tilde{S} := \{\xi \in S : \xi \text{ is finite}\}$. Show that the density of the upper invariant law of $(\xi_t)_{t\geq 0}$ is strictly positive, if and only if

$$\mathbb{P}(\tilde{\xi}_t^0 \neq \varnothing \, \forall t \geq 0) > 0.$$

Exercise 5 (20 pts)
Define for all $t \geq 0$

$$r_t := \sup \xi_t^0,$$

with the convention that $\sup \varnothing = -\infty$. Show that there exist constants $c, C > 0$ such that

$$\mathbb{P}(r_t > Ct) \leq e^{-ct} \quad \forall\, t \geq 0.$$

(In the solution you may use without proof that the moment generating function of an exponential random variable X with parameter λ is $\mathbb{E}[e^{sX}] = \lambda/(\lambda - s)$ for all $s < \lambda$.)